Example 1. Suppose

$\vec{F}$ is conservative, i.e.,

$\vec{F} = \nabla f$ for some $f: \mathbb{R}^2 \to \mathbb{R}$

Then $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$,

and by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Therefore, if R is <u>any</u> region with boundary $C = \partial R$, then

$$\int_C P\,dx + Q\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) dA$$

$$= 0$$

Let's use this result to calculate

$$\int_C 4x^3y\,dx + x^4\,dy$$

where C is the curve parameterized by $r(t) = \langle t, 1+\cos(t) \rangle$ for $0 \leq t \leq 2\pi$.

This is a nasty integral

But using Green's Theorem, if

$$\vec{F}(x,y) = \langle \underbrace{4x^3y}_{P(x,y)}, \underbrace{x^4}_{Q(x,y)} \rangle,$$

then

$$\frac{\partial P}{\partial y} = 4x^3 = \frac{\partial Q}{\partial x}, \text{ so}$$

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R (4x^3 - 4x^3)\, dA$$

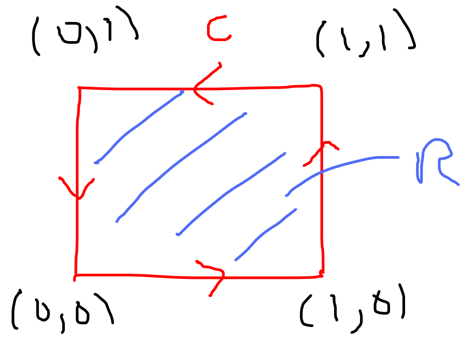
$$= \boxed{0}$$

Example 2

Let C be the square with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, traversed counterclockwise

If $\vec{F}(x,y) = \langle \underbrace{\cos(\pi(x+y))}_{P(x,y)}, \underbrace{e^{xy}}_{Q(x,y)} \rangle$,

find $\int_C \vec{F} \cdot d\vec{r}$



This would take four line integrals, but using Green's Theorem, we integrate

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where $R = [0,1] \times [0,1]$.

Now

$$\frac{\partial P}{\partial y} = -\pi \sin(\pi(x+y))$$

and

$$\frac{\partial Q}{\partial x} = y e^{xy}.$$

Then $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \int_0^1 \int_0^1 \left(y e^{xy} + \pi \sin(\pi(x+y))\right) dx\, dy$$

$$= \int_0^1 \int_0^1 y e^{xy} dx\, dy$$

$$+ \int_0^1 \int_0^1 \pi \sin(\pi(x+y)) dx\, dy$$

$\underline{1^{\text{st}}\text{ Integral}}$ Switch the order

$$\int_0^1 \int_0^1 y e^{xy}\, dy\, dx$$

$$= \int_0^1 \left(e^{xy} \Big|_0^1 \right) dx$$

$$= \int_0^1 (e^x - 1)\, dx$$

$$= (e^x - x)\Big|_0^1$$

$$= \boxed{e - 2}$$

2^{nd} Integral

$$\int_0^1 \int_0^1 \pi \sin(\pi(x+y))\, dx\, dy$$

$$= -\int_0^1 \left(\cos(\pi(x+y)) \Big|_0^1 \right) dy$$

$$= -\int_0^1 \left(\cos(\pi(y+1)) - \cos(\pi y) \right) dy$$

$$= \int_0^1 \left(\cos(\pi y) - \cos(\pi(y+1)) \right) dy$$

$$= \left(\frac{\sin(\pi y)}{\pi} - \frac{\sin(\pi(y+1))}{\pi} \right) \Big|_0^1$$

$$= \boxed{0}$$

Therefore,

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= e+2-0 = \boxed{e+2}$$

Example 3:

Calculate the area inside the region R bounded by the curve C parameterized by
$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$.

We know from general theory that

$$\iint_R 1 \, dA = \text{area of } R.$$

But how to find bounds for R?

Instead, let's use Green's Theorem and observe that if

$$\vec{F}(x,y) = \langle \underbrace{0}_{P}, \underbrace{x}_{Q} \rangle,$$

then $\dfrac{\partial P}{\partial y} = 0$ and $\dfrac{\partial Q}{\partial x} = 1$.

Then

$$\iint_R 1 \, dA$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_C \vec{F} \cdot d\vec{r} \quad \text{by Green's theorem}$$

Since we already have a parameterization for C, this is something we can evaluate

Note: There is no potential for $\vec{F}$, so we can't use the fundamental theorem for line integrals.

Brute force, then!

$$\vec{F}(\vec{r}(t)) = \langle 0, \cos^3(t) \rangle$$

$$\vec{r}\,'(t) = \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle$$

and the bounds should be from $t = 0$ to $t = 2\pi$.

Then

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle 0, \cos^3(t) \rangle \cdot \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle \, dt$$

$$= \int_0^{2\pi} 3\sin^2(t)\cos^4(t)\, dt$$

NOW

$$\int_0^{2\pi} 3\sin^2(t)\cos^4(t)\,dt \quad = \left(\cos^2(t)\right)^2$$

$$= 3\int_0^{2\pi} \left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right)^2 dt$$

(double-angle)

$$= \frac{3}{8}\int_0^{2\pi} \left(1-\cos^2(2t)\right)\left(1+\cos(2t)\right)dt$$

$$= \frac{3}{8}\int_0^{2\pi} \left(1-\cos^2(2t)-\cos^3(2t)+\cos(2t)\right)dt$$

Let's handle this as
3 separate integrals

1) $\int_0^{2\pi} (1+\cos(2t))\,dt$

$= \left(t + \frac{\sin(2t)}{2}\right)\Big|_0^{2\pi}$

$= \boxed{2\pi}$

2) $-\int_0^{2\pi} \cos^2(2t)\,dt$

$$= -\int_0^{2\pi} \frac{1+\cos(4t)}{2}\,dt$$

$$= -\left(\frac{t}{2} + \frac{\sin(4t)}{8}\right)\Bigg|_0^{2\pi}$$

$$= \boxed{-\pi}$$

3) $-\int_0^{2\pi} \cos^3(2t)\,dt$

$$= -\int_0^{2\pi} \cos(2t)(1-\sin^2(2t))\,dt$$

$$= -\int_0^{2\pi} \cos(2t)\,dt + \int_0^{2\pi} \cos(2t)\sin^2(2t)\,dt$$

$$= -\frac{\sin(2t)}{2}\Big|_0^{2\pi} + \frac{\sin^3(2t)}{6}\Big|_0^{2\pi}$$

$$= \boxed{0}$$

So the value of the integral is

$$2\pi - \pi = \boxed{\pi}$$

Multiplying by $\frac{3}{8}$, we get

$$\boxed{A = \frac{3\pi}{8}}$$

Area Equivalences

By using the vector fields

$\vec{F_1}(x,y) = \langle 0, x \rangle$, $\vec{F_2}(x,y) = \langle y, 0 \rangle$

and $\vec{F_3}(x,y) = \langle -\frac{y}{2}, \frac{x}{2} \rangle$,

respectively, we get

$$\boxed{\begin{aligned} A(R) = \int_R 1 \, dA &= \int_C x \, dy \\ &= -\int_C y \, dx \\ &= \frac{1}{2} \int x \, dy - y \, dx \end{aligned}}$$

Divergence and Curl

Now we consider vector fields on $\mathbb{R}^3$. The reason we define the subsequent terms is so we'll have better notation for the more mature fundamental theorem of calculus

Definition (curl)

If $\vec{F}(x,y,z) =$

$\langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

is a vector field on $\mathbb{R}^3$,

define

$$\boxed{\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle}$$

provided the partials exist

Fun notation

Define the ∇ ("del") operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

Then

$$\boxed{\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}}$$

Define the divergence of $\vec{F}$ to be

$$\boxed{\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Then

$$\boxed{\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}}$$

Simple yet profound

If $P, Q,$ and R have continuous second order partials, then if

$\vec{F} = \langle P, Q, R \rangle$,

$$\boxed{\operatorname{div}(\operatorname{curl}(\vec{F})) = 0}$$

Example 4. Compute the divergence and curl of

$$\vec{F}(x,y,z) = \langle x^2 y, \ln(xz), \arctan(xz^3) \rangle$$

Verify $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$

$$\operatorname{div}(\vec{F}) = 2xy + \frac{3z^2 x}{1 + x^2 z^6}$$

curl$(\vec{F})$

$$= \left\langle -\frac{1}{z},\ -\frac{z^3}{1+x^2z^6},\ \frac{1}{x} - x^2 \right\rangle$$

div(curl$(\vec{F})$)

$$= 0 + 0 + 0 = \boxed{0}$$